Size of Neutron Stars and White Dwarfs

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Abstract

In this paper we explore the ways in which energy due to gravity and due to Fermi gas contribute to the unique dimensions of Neutron Stars and White Dwarfs. We found that the minimum radius of a Neutron Star is approximately 10 km and the minimum radius of a White Dwarf is approximately 8,000 km.

1 Introduction

White Dwarfs and Neutron Stars are great examples of extremely dense collections of matter in the universe. Both are formed by the collapse of stars. The density and pressure found in both these stellar objects is much greater than anything that could be reproduced in a laboratory, so the measurements found by studying these objects provides information about physics fundamentals within such extreme environments.

White Dwarfs are formed by the collapse of low-mass stars and are composed of electron degenerate matter. They are the left-over core remnants of low mass stars that no longer sustain nuclear fusion. Within the layers of a main sequence star, Hydrogen is fused into Helium, and then Helium is fused into Carbon. Increased pressure caused by the nuclear fusion inside the star causes layers to expand, and then the star becomes unstable and bursts. White Dwarfs are mostly carbon and are very hot, but not hot enough to fuse carbon into anything else. Without any more nuclear fusion happening inside the star to create an outward pressure, electron degeneracy pressure is what stops White Dwarfs from gravitational collapse. The maximum mass of a white dwarf is known as the Chandrasekhar limit, which is $1.44M_{\odot}$.

The next densest collection of matter after a White Dwarf is a Neutron Star. Neutron Stars are essentially very dense balls of neutrons. The only stellar object more dense would be a black hole. Neutron stars result from the collapse of massive stars, and due to their high mass, the compression due to gravity surpasses that of White Dwarfs. In Neutron Stars, electrons and protons get fused together creating neutrons. Neutron degeneracy pressure is what keeps Neutron Stars from collapsing even further, as neutrons cannot be compressed indefinitely.

In this paper we aim to explain the mass-density ratio of White Dwarfs and Neutron Stars by breaking down the various aspects of density and pressure, and providing calculations for the pressure due to factors such as gravity and Fermi gas. We will also utilize phase space to discuss Fermi level and provide equations regarding the number of states or particles in each object.

2 Theory Outline

We are aiming to find the minimum radius for White Dwarf and Neutron stars, as Pauli's principle already tells us that only so many particles can fit into a given quantum mechanical state.[1] For simplicity, all of our equations will be made with the assumption that both White Dwarfs and Neutron Stars are uniform spherical distributions of matter.

$$E_{Gravity}(R,M) + E_{Fermi-gas}(R,M) = E_{total}(R,M) \tag{1}$$

To find the total energy of either a White Dwarf of a Neutron star, we must sum the energy due to gravity and the energy due to fermi gas. Both energies are a function of radius and mass. Pressure as a function of mass and radius is particularly interesting as both these stellar objects have limits in terms of how dense they can be before they release energy and become another object. A smaller radius results in higher pressure, and if the radius continues to shrink, the pressure grows and eventually results in explosion.

2.1 Energy due to Gravity

The gravitational energy as a function of the radius R and mass M can be written as

$$E_{\text{Gravity}} = -G\frac{3M^2}{5R}, \tag{2}$$

where $G = 6.67\times10^{-11}\, Nm^2/kg^2$ is the universal gravitational constant. The 3/5 constant is because we are assuming both White Dwarfs and Neutron Stars are of uniform spherical distribution.

2.2 Energy due to Fermi-gas

Fermi gas is the gas of fermions, which are identical particles of half-integer spin that obey the Pauli principle. Spin is the angular momentum of elementary particles. Examples of fermions include neutrons, electrons, and protons. In order to find the momentum due to Fermi gas, we derived an equation to describe this momentum as a function of the stellar object's radius and number of particles.

The number of particles can be found by dividing the mass of the particle in question by the mass of the object. For Neutron Stars, we take the mass of the star and divide it by the number of neutrons. For White Dwarfs, we do the same but for electrons.

$$N = \int 2\frac{dV dV_p}{(2\pi\hbar)^3} \tag{3}$$

$$N = \frac{32\pi^2}{9}\frac{R^3 P_F^3}{(2\pi\hbar)^3} \tag{4}$$

We can use Eq.(4) to find the Fermi momentum of the star where R is the radius of the star, N is the number of particles in the star, $\hbar = 1.055\times10^{-34} Js$ is the reduced Planck constant. The Planck constant, notated as h, is fundamental to quantum mechanics. The reduced Planck constant is used because we are describing the momentum of particles on an atomic scale, specifically the behavior of neutrons and electrons.

Momentum due to Fermi gas can be written as a function of the number of particles (Eq. 5).

$$P_F = \sqrt[3]{\frac{9\pi}{4}}\frac{\hbar}{R}N^{1/3}, \tag{5}$$

Phase space helps us understand why radius is an essential component of energy in this circumstance. Phase space consists of all possible states of a system. In phase space, particles do not occupy random space, but rather are placed directly next to each other in a uniform distribution.[2] The Pauli exclusion principle is a quantum mechanical principle that states two or more identical Fermions cannot occupy the same quantum mechanical state. This condition results in pressure because the matter is being compressed so severely.

We are mainly concerned with momentum and position, as our P_F will represent the maximum possible momentum of particles in the star. The Fermi level is defined by the number of particles to fit in a phase space.

As shown in the equation, a smaller radius will result in more pressure. Because we have a maximum number of particles that can fit in a given space, this results in increasing pressure inside the star. Thus, we will have a maximum energy level for each star.

The final equation for the energy due to Fermi gas as a function of radius and number of particles can be seen in Eq.(6).

$$E_{Fermi-gas} = \frac{9}{20}\sqrt[3]{\frac{3\pi^2}{2}}\frac{\hbar^2 N^{\frac{5}{3}}}{mR^2} \tag{6}$$

3 Results

We combined energy due to gravity and energy due to Fermi-gas to get the total energy of a star. The energy due to gravity is a function of radius and mass, whereas the energy due to Fermi-gas is a function of radius and the number of particles in the star. For Neutron Stars, this is neutrons. For White Dwarfs, this is electrons.

For a total energy (1), we get

$$E_{Total} = -G\frac{3M^2}{5R} + \frac{9}{20}\sqrt[3]{\frac{3\pi^2}{2}} \cdot \frac{\hbar^2 N^{\frac{5}{3}}}{mR^2}. \tag{7}$$

Looking back at Eq. (1), we see that our total energy is the sum of the two energies. While Eq. (7) could be simplified, for the purpose of better visualizing the combination of energies we leave it as is.

We created a plot for Neutron Stars and White Dwarfs using Eq. (7).

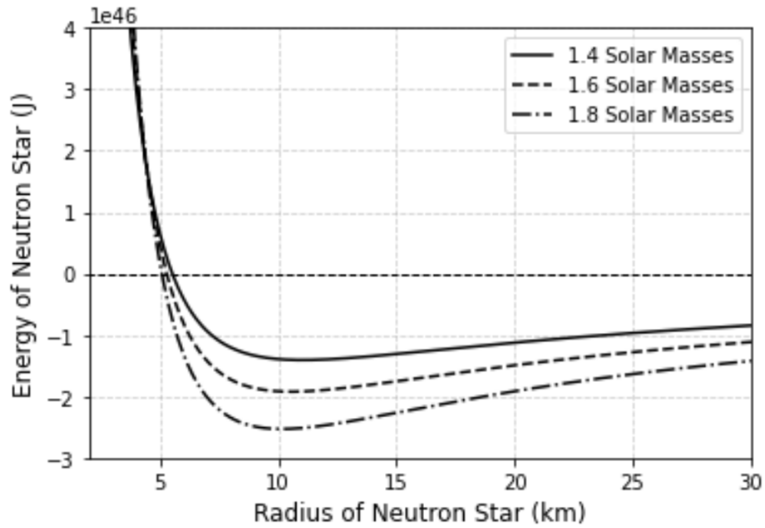

Figure 1: Total energy of Neutron Star as a function of the star radius (assuming different masses)

Figure 1 shows the total energy of three Neutron Stars with different solar masses as a function of radius. Looking at the minimum energy for each star, we can see that their radius is around the range of 10km.

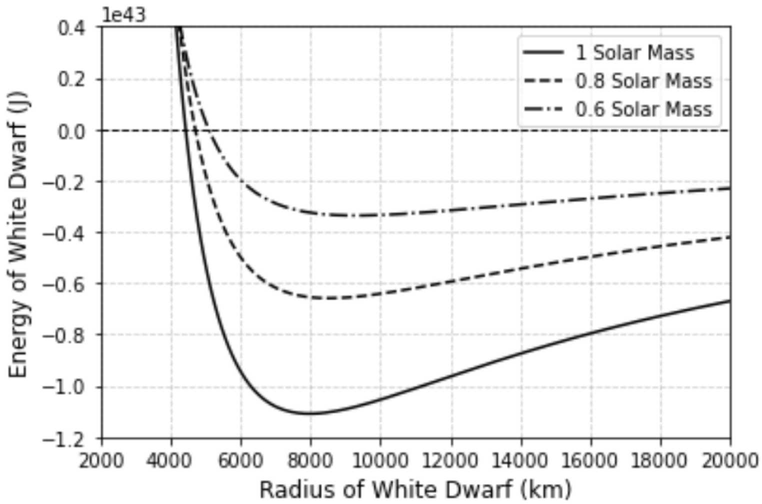

Figure 2: Total energy of White Dwarf star as a function of the star radius (assuming different masses)

Similarly, Figure 2 shows the total energy of three White Dwarfs with different masses as a function of radius. Looking at the minimum energy for each star, we see that their radius is around 8000km.

These graphs show that the radius of the star is a major component of the overall energy.

We then looked at the equations for radius for Neutron Stars and White Dwarfs. The masses in each equation are different because the mass of the particle responsible for repulsion in Neutron Stars and White Dwarfs are different. The equation for the radius of a Neutron star is

$$R_{NS} = (\frac{9\pi}{4})^{\frac{2}{3}} \frac{\hbar^2}{G M_n^{\frac{8}{3}} M^{\frac{1}{3}}}, \tag{8}$$

where M_n is the mass of a neutron and M is the mass of the star. Neutron Stars are composed of almost entirely neutrons, the mass any other particles are negligible and thus not considered in our equation.

The equation for the radius of a White Dwarf can be written as

$$R_{WD} = [(\frac{9\pi}{4})^{\frac{2}{3}} \frac{1}{2^{\frac{5}{3}}}] \frac{\hbar^2}{M_e M_n^{\frac{5}{3}} G M^{\frac{1}{3}}}. \tag{9}$$

For White Dwarfs, the main difference in comparison to Neutron Stars is that we are considering the mass of electrons rather than neutrons when calculating the radius. White dwarfs contain carbon and oxygen, specifically Carbon-12, which is made of 6 protons, 6 neutrons, and 6 electrons. White Dwarfs are composed of electron degenerate matter, which contributes to the electron degeneracy pressure that keeps White Dwarfs from collapsing further into a Neutron Star or black hole.[3]

When we plugged the numbers on the radius equation for Neutron stars, assuming a star of $1.4M_{\odot}$, we get a radius of 11.025 kilometers, which is around the value of radius that we got from looking at the graph of energy of Neutron Star.

Similarly, when we plugged in the numbers on the radius equation for a White Dwarf, assuming a star of $1M_{\odot}$, we get a radius of 7143 kilometers, which is around the value of radius that we got from looking at the graph of energy of White Dwarf.

References

[1] I. G. Kaplan. *The Pauli Exclusion Principle: Origin, Verifications, and Applications*. John Wiley & Sons, 2017.

[2] L. Cohen. "Generalized phase-space distribution functions". In: *Journal of Mathematical Physics* 7.5 (1966), pp. 781–786.

[3] D. Koester and G. Chanmugam. "Physics of white dwarf stars". In: *Reports on Progress in Physics* 53.7 (1990), p. 837.